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A theoretical scheme for generation of nonlinear coherent states in a micromaser under intensity-dependent Jaynes-Cummings model

M.H. Naderi[a], M. Soltanolkotabi, and R. Roknizadeh

Quantum Optics Group, Department of Physics, University of Isfahan, Isfahan, Iran

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Abstract. In this paper we propose a theoretical scheme to show the possibility of generating various families of nonlinear (f-deformed) coherent states of the radiation field in a micromaser. We show that these states can be provided in a lossless micromaser cavity under the weak Jaynes-Cummings interaction with intensity–dependent coupling of large number of individually injected two-level atoms in a coherent superposition of the upper and lower states. In particular, we show that the so-called nonlinear squeezed vacuum and nonlinear squeezed first excited states, as well as the even and odd nonlinear coherent states can be generated in a two-photon micromaser.

PACS. 42.50.Pq Cavity quantum electrodynamics; micromasers – 42.50.Dv Nonclassical states of the electromagnetic field, including entangled photon states; quantum state engineering and measurements

1 Introduction

The importance of coherent states (CSs) of various Lie algebras in different branches of physics, particularly in quantum optics, hardly needs to be emphasized. Historically, the conventional CSs of the quantum harmonic oscillator corresponding to the Heisenberg-Weyl algebra were first introduced by Schrödinger [1], who referred to them as states of minimum uncertainty product. The recognition that CSs are particularly important and appropriate for the quantum treatment of optical coherence and their adoption in quantum optics are due largely to the work of Glauber [2]. These states have quantum statistical properties like the classical radiation field and they define the limit between the classical and non-classical behaviors, like squeezing, antibunching and sub-Poissonian statistics. Subsequently the notion was generalized in various ways. Motivations to generalize the concept have arisen from symmetry considerations [3,4], dynamics [5] and algebraic aspects [6].

Recently a generalized class of the conventional CSs called the nonlinear coherent states (NLCSs) [7] or the f-CSs [8], which can be classified as an algebraic generalization of the conventional CSs, have been constructed. These states, which correspond to nonlinear algebras rather than Lie algebras, are defined as right eigenstates of the generalized annihilation operator $\hat{A} = \hat{a} f(\hat{N})$,

$$\hat{A}\,|z\rangle_f = z\,|z\rangle_f\,, \tag{1}$$

where $f(\hat{N})$ is a reasonably well-behaved real function of the number operator $\hat{N} = \hat{a}^{+}\hat{a}$ and z is an arbitrary complex number. From (1) one can obtain an explicit form of the NLCSs in a number state representation

$$|z\rangle_f = C\sum_{n=0}^{\infty} z^n d_n\,|n\rangle\,, \tag{2}$$

where the coefficients d_n's and normalization constant C, respectively, are given by

$$d_0 = 1,\quad d_n = \left(\sqrt{n!}f(n)!\right)^{-1},\quad f(n)! \equiv \prod_{j=1}^{n} f(j) \tag{3}$$

$$C = \left(\sum_{n=0}^{\infty} d_n^2 |z|^{2n}\right)^{-1/2}. \tag{4}$$

Actually, NLCSs have been known for many years under other names. The phase state [9] or its generalization [10] (known nowadays as the negative binomial state or the $SU(1,1)$ group coherent states) and the photon-added CS [11] are two well-known examples of NLCSs. The physical meaning of NLCSs has been elucidated in [7,8],

[a] e-mail: mhnaderi2001@yahoo.com

where it has been shown that such states may appear as stationary states of the center-of-mass motion of a trapped ion [7], or may be related to some nonlinear processes (such as a hypothetical "frequency blue shift" in high intensity photon beams [8]). Furthermore, it has been shown that NLCSs exhibit various non-classical features such as quadrature squeezing, number-phase squeezing and sub-Poissonian photon statistics [12].

On the other hand, in last few years the production and detection of non-classical states of the radiation field have attracted a great deal of attention because of their latent applications in optical communication and in precise and sensitive measurements. One of the marvelous experimental and theoretical systems which can be utilized to produce non-classical radiation is the one–atom maser or micromaser [13]. The system consists of a high-Q microwave cavity and a stream of injected Rydberg atoms which drive the field inside the cavity. The atomic beam is sufficiently sparse so that no more than one atom is in the cavity at any time. There are several schemes that have been proposed to produce number states [14], sub-Poissonian states [15], squeezed states [16] and trapping states [17] and the possibility of generating pure states of the field, the so-called tangent and cotangent states [18] has also been predicted by using micromasers.

Although foregoing schemes have been discussed by utilizing the standard Jaynes-Cummings model [19], one may consider the generalized multi-photon intensity-dependent Jaynes-Cummings model (IDJC). The multi-photon IDJC model is a quantum model describing the interaction of a monochromatic electromagnetic field with one two-level atom in a cavity under intensity-dependent coupling through multi-photon transitions. The interaction Hamiltonian of this model can be expressed in the rotating-wave approximation and in the interaction picture as following ($\hbar = 1$)

$$\hat{H}_{ID}^{(m)} = g\left(\hat{a}^m f(\hat{N})\left|a\right\rangle\left\langle b\right| + \left|b\right\rangle\left\langle a\right| f(\hat{N})(\hat{a}^+)^m\right); \quad (m = 1, 2, ...). \quad (5)$$

Here, $\hat{a}$ and $\hat{a}^+$ are the usual operators of the Heisenberg-Weyl algebra, $[\hat{a}, \hat{a}^+] = 1$, $|a\rangle$, $|b\rangle$ are the excited and ground atomic states, respectively and g is the coupling constant. The function $f(\hat{N})$, which is assumed to be real, describes the intensity dependence of atom-field interaction. Some particular forms of IDJC have been studied in literature. For example, the model equation (5) with $f(\hat{N}) = \sqrt{\hat{N}}$ and $m = 1$ was first introduced in [20] and later analyzed in [21]. In particular, this type of IDJC model is interesting because of its inherent connection to an $SU(1,1)$ Jaynes-Cummings model. A q-boson generalization of this model was introduced in [22].

Admittedly the intensity-dependent coupling between the field and the atom requires further justification. In this respect it is worth mentioning that such a justification is required for any quantum-optical model based on the two-level approximation. The Hamiltonian describing the interaction of the two-level system with the quantized field mode should be understood as "effective". This means that only two atomic levels are effectively singled out from the energy spectrum. This operator includes two field operators related to the ground and excited states in question as well as operator multiplier related to the dipole transition. These operators are expressed in terms of the Bose operators of the quantized field mode in a very complicated way. They should account for various Stark shifts of the quantum levels as well as for the dependence of the dipole moment of the quantum transition on the state of the exciting field. It is possible in principle to explain how to construct these effective operators for any atomic Hamiltonian and for any pair of quantum levels using Kato's transformation operator (cf. [23]) and the secular operator of degenerate perturbation theory. It is also worth mentioning that generalized Jaynes-Cummings models have recently become the subject of intense attention [24–30]. These considerations support the theoretical interest in the IDJC model since this kind of interaction means effectively that the coupling constant is proportional to the intensity of the cavity field which represents a very simple case of a nonlinear interaction corresponding to a more realistic physical situation. Moreover, it can potentially provide various variants of the field state possessing interesting quantum statistical features. In other words, the model equation (5) may be considered as a useful theoretical laboratory in which time evolution of a variety of initial states of the system can be analyzed.

On the other hand, experiments of increasing difficulty in cavity quantum electrodynamics over the last years have made it possible to test fundamental radiation–matter interaction models involving single atoms [31]. Such a stimulating situation essentially stems from two decisive advancements. The first is the invention of reliable protocols for the manipulation of single atoms. The second is the ability to produce desired bosonic environments on demand. This progress has led to the possibility of controlling the form of the coupling between individual atoms and an arbitrary number of bosonic modes. As a consequence, fundamental matter–radiation interaction models such as the Jaynes-Cummings model and most of its numerous nonlinear generalizations, have been realized or simulated in the laboratory and their dynamical features have been tested more or less in detail.

In the present contribution, we aim at exploring the possibility of generating various families of NLCSs in a micromaser under IDJC model described by the model equation (5). We find that they occur under the conditions that cavity losses are negligible (a possible situation in the micromaser experiments [13]) and the injected two-level atoms are prepared in a coherent superposition of the upper and lower states. By considering the weak interaction of large number of individual atoms with the cavity-field through one as well as two-photon transitions we investigate the quantum evolution of the cavity-field state. In the next section we are dealing with the case of one-photon transitions and Section 3 is devoted to the case of two-photon transitions. The results are summarized in Section 4.

2 Quantum evolution of the cavity-field state: one-photon transitions

We consider a beam of monoenergetic two-level atoms injected at regular time intervals into a lossless microwave cavity in which the atoms interact resonantly with the cavity mode for a finite time τ. The micromaser is usually operated in the regime in which there is at most only one atom in the cavity at any time. It is assumed that the time of the interaction of each atom with the cavity-field is much shorter than the lifetime of all the atomic levels. Then the atomic spontaneous decay processes to other levels can be ignored while an atom is inside the cavity, which means that the joint evolution of the cavity-field and atoms is unitary.

We assume that injected atoms interact with the cavity-field through one-photon transitions and intensity-dependent coupling. The Hamiltonian describing the atom-field interaction is given by model equation (5) with $m = 1$ that may be written as

$$\hat{H}_{ID}^{(m=1)} = g\left(\hat{A}\left|a\right\rangle\left\langle b\right| + \left|b\right\rangle\left\langle a\right|\hat{A}^{+}\right), \tag{6}$$

in which

$$\hat{A} = \hat{a}f(\hat{N}), \quad \hat{A}^{+} = f(\hat{N})\hat{a}^{+}. \tag{7}$$

It is assumed that the real function $f(\hat{N})$is such that it has no zeros at positive integer values of n, including zero. From the relations (7) it follows that $\hat{A}$, $\hat{A}^{+}$ and the number operator $\hat{N}$ satisfy the following closed algebraic relations

$$[\hat{A}, \hat{A}^{+}] = \{\hat{N}+1\}_f - \{\hat{N}\}_f,$$
$$[\hat{N}, \hat{A}] = -\hat{A}, \quad [\hat{N}, \hat{A}^{+}] = \hat{A}^{+}, \tag{8a}$$

together with

$$\hat{A}\left|n\right\rangle = \sqrt{\{n\}_f}\left|n-1\right\rangle,$$
$$\hat{A}^{+}\left|n\right\rangle = \sqrt{\{n+1\}_f}\left|n+1\right\rangle, \quad \hat{N}\left|n\right\rangle = n\left|n\right\rangle, \tag{8b}$$

where the symbol $\{X\}_f$ stands for $Xf^2(X)$. Thus the relations (8a) represent a deformed Heisenberg algebra whose nature of deformation depends on the nonlinearity function $f(\hat{N})$. Clearly for $f(\hat{N})$ =1 we regain the Heisenberg algebra.

It is easy to show that the corresponding time evolution operator $\hat{U}(\tau)$ can be expressed in the form

$$\begin{aligned}\hat{U}(\tau) \equiv \exp(-i\hat{H}_{ID}^{(m=1)}\tau) = {} & \cos\left(g\tau\sqrt{\{\hat{N}+1\}_f}\right)\left|a\right\rangle\left\langle a\right| \\ & + \cos\left(g\tau\sqrt{\{\hat{N}\}_f}\right)\left|b\right\rangle\left\langle b\right| \\ & - i\frac{\sin\left(g\tau\sqrt{\{\hat{N}+1\}_f}\right)}{\sqrt{\{\hat{N}+1\}_f}}\hat{A}\left|a\right\rangle\left\langle b\right| \\ & - i\frac{\sin\left(g\tau\sqrt{\{\hat{N}\}_f}\right)}{\sqrt{\{\hat{N}\}_f}}\hat{A}^{+}\left|b\right\rangle\left\langle a\right|.\end{aligned} \tag{9}$$

Let all atoms are initially prepared in a same superposition of the upper level $|a\rangle$ and the lower level $|b\rangle$. Therefore the initial density matrix of the Kth atom can be written as

$$\hat{\rho}_A^{(K)}(t=0) = \sum_{i,j=a,b}\rho_{ij}\left|i_K\right\rangle\left\langle j_K\right|, \tag{10}$$

where $\rho_{ii} \geq 0$, $\rho_{\rm aa} + \rho_{\rm bb} = 1$, $\rho_{\rm ab} = \rho_{ba}^{*} = |\rho_{\rm ab}|\exp(i\varphi)$, $|\rho_{\rm ab}| = |\rho_{\rm ba}| \equiv \sqrt{\rho_{aa}\rho_{bb}}$.

It should be noted that there is a free evolution of the cavity-field density matrix in the time between the subsequent atoms entering the cavity, i.e., the matrix elements of the cavity-field density matrix acquire an extra phase factor $\exp(i(n-n')\omega\delta t)$, where ω is the cavity resonance frequency and δt is the time between the arrivals of subsequent atoms. We assume here that the time δt is chosen in such a way that $\omega\delta t$ is equal to a multiple of 2π. In this case the extra phase factor due to the free evolution is unity. Otherwise we should take it into account in the overall density matrix evolution. If the atoms were arriving at random times they would meet the cavity-field with random phases, and the cavity-field phase, which is associated with the non-diagonal elements of the density matrix, would necessarily become random (only diagonal elements would survive). This assumption is a very serious restriction to the model considered here. It means that atoms should be injected into the cavity in a well controllable way.

Assuming that it is possible, the field density matrix, after passing K atoms through the micromaser cavity reads as

$$\hat{\rho}_F^{(K)} = \mathrm{Tr}_A\left(\hat{U}_K(\tau)\hat{\rho}_A^{(K)} \otimes \hat{\rho}_F^{(K-1)}\hat{U}_K^{+}(\tau)\right), \tag{11}$$

in which Tr_A indicates partial trace over the Hilbert space of the two-level atom. Here, the number of injected atoms K is considered as a scaled evolution time of the system. By using (11) together with the expressions (9) and (10), we can easily get for the cavity-field density matrix elements the recursion relation

$$
\begin{aligned}
\rho_F^{(K)}(n,n') &= \langle n|\,\hat{\rho}_F^{(K)}\,|n'\rangle = \\
&\Bigg(\rho_{aa}\cos\left(g\tau\sqrt{\{n+1\}_f}\right)\cos\left(g\tau\sqrt{\{n'+1\}_f}\right) \\
&+\rho_{bb}\cos\left(g\tau\sqrt{\{n\}_f}\right)\cos\left(g\tau\sqrt{\{n'\}_f}\right)\Bigg)\rho_F^{(K-1)}(n,n') \\
&+\rho_{bb}\sin\left(g\tau\sqrt{\{n+1\}_f}\right)\sin\left(g\tau\sqrt{\{n'+1\}_f}\right) \\
&\quad\times\rho_F^{(K-1)}(n+1,n'+1) \\
&+\rho_{aa}\sin\left(g\tau\sqrt{\{n\}_f}\right)\sin\left(g\tau\sqrt{\{n'\}_f}\right) \\
&\quad\times\rho_F^{(K-1)}(n-1,n'-1) \\
&+i|\rho_{ab}|\exp(i\varphi)\cos\left(g\tau\sqrt{\{n+1\}_f}\right) \\
&\quad\times\sin\left(g\tau\sqrt{\{n'+1\}_f}\right)\rho_F^{(K-1)}(n,n'+1) \\
&+i|\rho_{ab}|\exp(-i\varphi)\cos\left(g\tau\sqrt{\{n\}_f}\right)\sin\left(g\tau\sqrt{\{n'\}_f}\right) \\
&\quad\times\rho_F^{(K-1)}(n,n'-1) \\
&-i|\rho_{ab}|\exp(-i\varphi)\sin\left(g\tau\sqrt{\{n+1\}_f}\right) \\
&\quad\times\cos\left(g\tau\sqrt{\{n'+1\}_f}\right)\rho_F^{(K-1)}(n+1,n') \\
&-i|\rho_{ab}|\exp(i\varphi)\sin\left(g\tau\sqrt{\{n\}_f}\right)\cos\left(g\tau\sqrt{\{n'\}_f}\right) \\
&\quad\times\rho_F^{(K-1)}(n-1,n'),
\end{aligned}
\tag{12}
$$

with the initial condition $\rho_F^{(K=0)}(n,n') = \rho_F^{(0)}(n,n')$. It is seen from the recursion relation (12) that the coupling between the diagonal matrix elements $\rho_F^{(K)}(n,n)$ and the off-diagonal elements $\rho_F^{(K)}(n,n\pm1) = \rho_F^{(K)^*}(n\pm1,n)$ occurs only when the atomic coherence ρ_{ab} is present. If the micromaser is pumped by unpolarized atoms ($\rho_{ab}=0$) then the off-diagonal elements don't occur, and consequently the field phase is always random. However, atoms prepared in a coherent superposition of their states before entering the micromaser cavity create nonvanishing off-diagonal elements, that is they create a preferred phase field [14,32].

In order to solve the recursion relation (12) we adopt a method which has firstly been used by Kien et al. [33]. We introduce the phase-independent matrix elements $\tilde{\rho}^{(K)}(n,n')$ through the definition

$$
\rho_F^{(K)}(n,n') = (i\exp(-i\varphi))^{n'-n}\left(\prod_{\ell=1}^{n}\sin\left(g\tau\sqrt{\{\ell\}_f}\right)\times\prod_{\ell'=1}^{n'}\sin\left(g\tau\sqrt{\{\ell'\}_f}\right)\right)\rho_{aa}^{(n+n')/2}\tilde{\rho}^{(K)}(n,n'). \tag{13}
$$

Substituting this expression into (12), we get the recursion relation

$$
\begin{aligned}
\tilde{\rho}^{(K)}(n,n') &= \\
&\Bigg(\rho_{aa}\cos\left(g\tau\sqrt{\{n+1\}_f}\right)\cos\left(g\tau\sqrt{\{n'+1\}_f}\right) \\
&+\rho_{bb}\cos\left(g\tau\sqrt{\{n\}_f}\right)\cos\left(g\tau\sqrt{\{n'\}_f}\right)\Bigg)\tilde{\rho}^{(K-1)}(n,n') \\
&+\rho_{aa}\rho_{bb}\sin^2\left(g\tau\sqrt{\{n+1\}_f}\right)\sin^2\left(g\tau\sqrt{\{n'+1\}_f}\right) \\
&\quad\times\tilde{\rho}^{(K-1)}(n+1,n'+1)+\tilde{\rho}^{(K-1)}(n-1,n'-1) \\
&-\rho_{aa}\sqrt{\rho_{bb}}\cos\left(g\tau\sqrt{\{n+1\}_f}\right)\sin^2\left(g\tau\sqrt{\{n'+1\}_f}\right) \\
&\quad\times\tilde{\rho}^{(K-1)}(n,n'+1) \\
&-\rho_{aa}\sqrt{\rho_{bb}}\sin^2\left(g\tau\sqrt{\{n+1\}_f}\right)\cos\left(g\tau\sqrt{\{n'+1\}_f}\right) \\
&\quad\times\tilde{\rho}^{(K-1)}(n+1,n') \\
&+\sqrt{\rho_{bb}}\cos\left(g\tau\sqrt{\{n\}_f}\right)\tilde{\rho}^{(K-1)}(n,n'-1) \\
&+\sqrt{\rho_{bb}}\cos\left(g\tau\sqrt{\{n'\}_f}\right)\tilde{\rho}^{(K-1)}(n-1,n'),
\end{aligned}
\tag{14}
$$

with the initial condition

$$
\tilde{\rho}^{(0)}(n,n') = (i\exp(-i\varphi))^{n-n'}\left(\prod_{\ell=1}^{n}\sin\left(g\tau\sqrt{\{\ell\}_f}\right)\times\prod_{\ell'=1}^{n'}\sin\left(g\tau\sqrt{\{\ell'\}_f}\right)\right)^{-1}\rho_{aa}^{-(n+n')/2}\rho_F^{(0)}(n,n'). \tag{15}
$$

Now we consider the case of weak atom-field interaction, that is,

$$
g\tau \ll 1, \quad g\tau\sqrt{\{\langle n\rangle\}_f} \equiv g\tau\sqrt{\langle n\rangle f^2(\langle n\rangle)} \ll 1, \tag{16}
$$

where $\langle n\rangle$ is the mean photon number of the cavity-field. In the first-order approximation that is,

$$
\sin\left(g\tau\sqrt{\{n\}_f}\right) \approx g\tau\sqrt{\{n\}_f},
$$

$$
\sin^2\left(g\tau\sqrt{\{n\}_f}\right) \approx 0, \quad \cos\left(g\tau\sqrt{\{n\}_f}\right) \approx 1
$$

the recursion relation (14) becomes

$$
\tilde{\rho}^{(K)}(n,n') \approx \tilde{\rho}^{(K-1)}(n,n') + \tilde{\rho}^{(K-1)}(n-1,n'-1) + \sqrt{\rho_{bb}}\left(\tilde{\rho}^{(K-1)}(n,n'-1) + \tilde{\rho}^{(K-1)}(n-1,n')\right). \tag{17}
$$

The solution of equation (17) is easily found to be

$$
\tilde{\rho}^{(K)}(n,n') \approx \sum_{k,k'=0}^{K}\sum_{p=0}^{\min(k,k')}\frac{K!\,\rho_{bb}^{(k+k'-2p)/2}}{p!(k-p)!(k'-p)!(K-k-k'+p)!}\times\tilde{\rho}^{(0)}(n-k,n'-k'). \tag{18}
$$

Since the expression (18) is not very convenient to be used for large K, we prefer to use the truncated form

$$\tilde{\rho}^{(K)}(n,n') \approx \sum_{k=0}^{n}\sum_{k'=0}^{n'}\sum_{p=0}^{\min(k,k')} \frac{K!\rho_{bb}^{(k+k'-2p)/2}}{p!(k-p)!(k'-p)!(K-k-k'+p)!} \times \tilde{\rho}^{(0)}(n-k,n'-k'). \quad (19)$$

Substituting equation (19) into equation (13), we obtain for $\rho_F^{(K)}(n,n')$ the approximate expression

$$\rho_F^{(K)}(n,n') \approx (i\exp(-i\varphi))^{n'-n}(g\tau\sqrt{\rho_{aa}})^{n+n'}\sqrt{\{n\}_f!\{n'\}_f!} \times \sum_{k=0}^{n}\sum_{k'=0}^{n'}\sum_{p=0}^{\min(k,k')} \frac{K!\rho_{bb}^{(k+k'-2p)/2}}{p!(k-p)!(k'-p)!(K-k-k'+p)!} \times \tilde{\rho}^{(0)}(n-k,n'-k'), \quad (20)$$

where by definition $\{n\}_f! = \{n\}_f\{n-1\}_f\{n-2\}_f...1$ and $\{0\}_f! = 1$. Now let $\rho_{bb} \neq 0$ and $K \gg 1$. The relation between $(p+1)$th and pth terms in the sum on the r.h.s of expression (19) is

$$\frac{(k-p)(k'-p)}{\rho_{bb}(p+1)(K-k-k'+1)} \leq \frac{kk'}{\rho_{bb}(K-k-k')}. \quad (21)$$

As it is seen, in the region of values of n and n' such that $n+n'+nn'/\rho_{bb} \ll K$ the term with $p=0$ in the sum on the r.h.s of (19) dominates. Keeping only the $p=0$ term and using the approximation $K!/(K-k-k')! \approx K^{(k+k')}$, from expression (20) we find

$$\rho_F^{(K)}(n,n') \approx \sqrt{\{n\}_f!\{n'\}_f!}\sum_{k=0}^{n}\sum_{k'=0}^{n'}(i\exp(-i\varphi))^{k'-k} \times \frac{(Kg\tau\sqrt{\rho_{aa}\rho_{bb}})^{k+k'}}{k!k'!\sqrt{\{n-k\}_f!\{n'-k'\}_f!}}\rho_F^{(0)}(n-k,n'-k'), \quad (22)$$

in which we have made use of (15). The expression (22) gives the matrix elements of the cavity-field density matrix after passing a large number of injected atoms through the micromaser cavity where each atom undergoes one-photon transitions under the weak atom-field interaction with intensity-dependent coupling. If the micromaser starts from a pure state $|\psi_F^{(0)}\rangle$, then the cavity-field evolves into the pure state $|\psi_F^{(K)}\rangle$ as follows

$$\left\langle n \,\middle|\, \psi_F^{(K)}\right\rangle \approx \sum_{k=0}^{n}\frac{z^k}{k!\sqrt{\{n-k\}_f!}}\sqrt{\{n\}_f!}\left\langle n-k \,\middle|\, \psi_F^{(0)}\right\rangle \quad (23)$$

with $z = -i\exp(-i\varphi)Kg\tau\sqrt{\rho_{aa}\rho_{bb}}$.

Now let us assume that the cavity-field is initially in the vacuum state, i.e., $|\psi_F^{(0)}\rangle = |0\rangle$. Thus from (23) we find the following approximate expression for the normalized cavity-field state after passing K atoms ($K \gg 1$)

$$\left|\psi_F^{(K)}\right\rangle \equiv |z\rangle_f' = C'\sum_{n=0}^{\infty}\frac{z^n}{n!}\sqrt{\{n\}_f!}\,|n\rangle = C'\sum_{n=0}^{\infty}z^n d_n'\,|n\rangle, \quad (24)$$

where

$$d_0' = 1, \quad d_n' = \left(\sqrt{n!}\big/f(n)!\right)^{-1}, \quad (25a)$$

and C' is the normalization constant given by

$$C' = \left(\sum_{n=0}^{\infty}d_n'^2|z|^{2n}\right)^{-1/2}. \quad (25b)$$

It is evident that for $f(n)=1$ the state vector (24) describes the usual CS. Therefore it is reasonable to interpret the state (24) as a NLCS of the cavity-field. But, since the expansion coefficients d_n' are different from the coefficients d_n, given by (3) (provided of course we use the same nonlinearity function $f(n)$ in both the cases) the NLCS obtained in (24) is distinct from the NLCS defined in equation (2).

In order to get more clear insight to the above result we present the following argument. From the relation (8a) we find that the r.h.s of the commutator $[\hat{A},\hat{A}^+]$ is a nonlinear function of the number operator $\hat{N}$. As a result BCH disentangling theorem [34] can not be applied and one can not use the displacement operator $\exp(z\hat{A}-z^*\hat{A}^+)$ to construct coherent states. Therefore one may seek for an operator $\hat{B}^+$ which is conjugate of the operator $\hat{A}$, that is $[\hat{A},\hat{B}^+]=1$ while their Hermitian conjugates $\hat{A}^+$ and $\hat{B}$ satisfy the dual algebra $[\hat{B},\hat{A}^+]=1$. From (7) it is easily found that

$$\hat{B} = \hat{a}\frac{1}{f(\hat{N})}, \quad \hat{B}^+ = \frac{1}{f(\hat{N})}\hat{a}^+. \quad (26)$$

Let us now consider the following displacement operators

$$\hat{D}_f(z) = \exp(z\hat{B}^+ - z^*\hat{A}), \quad \hat{D}'_f(z) = \exp(z\hat{A}^+ - z^*\hat{B}) \quad (27)$$

and note that for any two operators $\hat{X}$ and $\hat{Y}$ satisfying the relation $[\hat{X},\hat{Y}]=1$ the BCH theorem results in

$$\exp(z\hat{X}-z^*\hat{Y}) = \exp(-|z|^2/2)\exp(z\hat{X})\exp(-z^*\hat{Y}). \quad (28)$$

Now it is easy to find that the NLCSs (2) can be defined as $|z\rangle_f = \hat{D}_f(z)\,|0\rangle$, corresponding to the dual algebra $[\hat{B},\hat{A}^+]=1$ while the NLCSs given by (24) can be defined as $|z\rangle_f' = \hat{D}_f'(z)\,|0\rangle$, corresponding to the algebra $[\hat{A},\hat{B}^+]=1$. In addition, we have

$$\hat{A}\,|z\rangle_f = \hat{a}f(\hat{N})\,|z\rangle_f = z\,|z\rangle_f\,, \qquad \hat{B}\,|z\rangle_f' = \hat{a}\frac{1}{f(\hat{N})}\,|z\rangle_f' = z\,|z\rangle_f'\,. \quad (29)$$

In this manner, we conclude that the intensity-dependent Jaynes-Cummings Hamiltonian (6), under the conditions of weak atom-field interaction and the passage of large number of polarized atoms through the cavity initially prepared in vacuum state, results in the NLCS (24) which is the eigenstate of the deformed annihilation operator $\hat{B}$. While if one considers the intensity-dependent interaction Hamiltonian as $\hat{H}_{ID}^{(m=1)} = g(\hat{B}|a\rangle\langle b| + |b\rangle\langle a|\hat{B}^+)$ then under the same conditions the cavity-field evolves to the NLCS (2) which is the eigenstate of the operator $\hat{A}$.

It is to be noted that the normalizability of the states $|z\rangle_f$ and $|z\rangle'_f$ depends crucially on the convergence of the series which define the normalization constants C and C'. This means that the normalization constants should be nonzero and finite. This condition results in the following restrictions for the values of $|z|^2 = (Kg\tau)^2\rho_{aa}\rho_{bb}$ for the states $|z\rangle_f$ and $|z\rangle'_f$ respectively,

$$|z|^2 < \lim_{m\to\infty}(m+1)f^2(m+1), \tag{30a}$$

$$|z|^2 < \lim_{m\to\infty}\frac{(m+1)}{f^2(m+1)}. \tag{30b}$$

It is evident that if $f(m)$ increases (decreases) faster than $m(m^{-1})$ for large m, then in the case of $|z\rangle_f$ ($|z\rangle'_f$) the range of $|z|^2$ is unrestricted.

The mean number of photons $\langle n\rangle$ and the Mandel parameter Q, which measures the deviation from Poissonian statistics ($Q = 0$), for these two states are respectively given by

$$\langle n\rangle_{|z\rangle_f} = |z|^2\frac{\frac{\partial}{\partial|z|^2}C^{-2}}{C^{-2}}, \quad \langle n\rangle_{|z\rangle'_f} = |z|^2\frac{\frac{\partial}{\partial|z|^2}C'^{-2}}{C'^{-2}}, \tag{31a}$$

$$Q_{|z\rangle_f} = \frac{|z|^2}{\langle n\rangle_{|z\rangle_f}}\frac{\partial\langle n\rangle_{|z\rangle_f}}{\partial|z|^2} - 1,$$

$$Q_{|z\rangle'_f} = \frac{|z|^2}{\langle n\rangle_{|z\rangle'_f}}\frac{\partial\langle n\rangle_{|z\rangle'_f}}{\partial|z|^2} - 1. \tag{31b}$$

In Figure 1 we plot the photon-number distribution for the state $|z\rangle_f$ with $g\tau = 10^{-3}$, $K = 10^4$, $\rho_{aa} = \rho_{bb} = 0.5$ and for three different cases $f(n) = 1$ (linear micromaser), $f(n) = \sqrt{n+1}$, $f(n) = n + 1$. Using (31a) and (31b) we also find that

$$\langle n\rangle_{|z\rangle_{f=1}} = |z|^2 = 25; \quad Q_{|z\rangle_{f=1}} = 0 \quad \text{(Poissonian statistics)},$$

$$\langle n\rangle_{|z\rangle_{f=\sqrt{n+1}}} = 4.270; \quad Q_{|z\rangle_{f=\sqrt{n+1}}} = -0.417 \quad \text{(sub-Poissonian statistics)},$$

$$\langle n\rangle_{|z\rangle_{f=n+1}} = 2.281; \quad Q_{|z\rangle_{f=n+1}} = -0.576 \quad \text{(sub-Poissonian statistics)}.$$

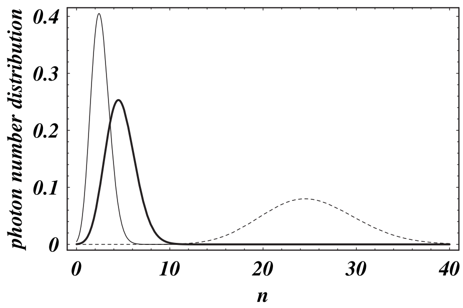


Fig. 1. Photon number distribution for the state $|z\rangle_f$ with $g\tau = 10^{-3}$, $K = 10^4$, $\rho_{aa} = \rho_{bb} = 0.5$ and for three different cases $f(n) = 1$ (------) $f(n) = \sqrt{n+1}$ (▬▬) and $f(n) = n + 1$ (——).

3 Quantum evolution of the cavity-field state: two-photon transitions

So far, we have discussed the possibility of the generation NLCSs in a micromaser in which the injected atoms interact with the cavity–field through one-photon transitions. Now we are intended to examine the problem for the case of two-photon transitions. The two-photon transitions are results of nonlinear atom-field interaction and are the higher order processes. Therefore, the probability of such transitions is extremely small when compared to the probability of one-photon transitions. Moreover, the higher order processes require longer atom-field interaction time since the atom-field coupling is drastically reduced and as a consequence the decoherence can occur during the interaction. Nevertheless, two-photon micromasers were both theoretically [35] and experimentally [36] studied.

The interaction Hamiltonian is given by model equation (5) with $m = 2$ that may be written as

$$\hat{H}_{ID}^{(m=2)} = g\left(\hat{C}\,|a\rangle\,\langle b| + |b\rangle\,\langle a|\,\hat{C}^+\right), \tag{32}$$

where

$$\hat{C} = \hat{a}^2 f(\hat{N}), \quad \hat{C}^+ = f(\hat{N})(\hat{a}^+)^2. \tag{33}$$

The operators $\hat{C}$, $\hat{C}^+$ and $\hat{N}$ satisfy the following closed algebraic relations

$$[\hat{C},\hat{C}^+] = [\hat{N}+2]_f - [\hat{N}]_f,$$
$$[\hat{N},\hat{C}] = -2\hat{C}, \quad [\hat{N},\hat{C}^+] = 2\hat{C}^+, \tag{34a}$$

together with

$$\hat{A}\,|n\rangle = \sqrt{[n]_f}\,|n-2\rangle\,, \quad \hat{A}^+\,|n\rangle = \sqrt{[n+2]_f}\,|n+2\rangle\,, \tag{34b}$$

in which the symbol $[X]_f$ stands for $X(X-1)f^2(X)$. It should be noted that the deformed operator $\hat{C}$ annihilates both the vacuum state $|0\rangle$ and first excited state $|1\rangle$.

In this case we obtain the following recursion relation for the cavity-field density matrix elements after passing

K polarized atoms through the micromaser cavity

$$\rho_F^{(K)}(n,n') = \langle n|\,\hat{\rho}_F^{(K)}\,|n'\rangle =$$
$$\Big(\rho_{aa}\cos\left(g\tau\sqrt{[n+2]_f}\right)\cos\left(g\tau\sqrt{[n'+2]_f}\right)$$
$$+\rho_{bb}\cos\left(g\tau\sqrt{[n]_f}\right)\cos\left(g\tau\sqrt{[n']_f}\right)\Big)\rho_F^{(K-1)}(n,n')$$
$$+\rho_{bb}\sin\left(g\tau\sqrt{[n+2]_f}\right)\sin\left(g\tau\sqrt{[n'+2]_f}\right)$$
$$\times\rho_F^{(K-1)}(n+2,n'+2)$$
$$+\rho_{aa}\sin\left(g\tau\sqrt{[n]_f}\right)\sin\left(g\tau\sqrt{[n']_f}\right)$$
$$\times\rho_F^{(K-1)}(n-2,n'-2)$$
$$+i|\rho_{ab}|\exp(i\varphi)\cos\left(g\tau\sqrt{[n+2]_f}\right)\sin\left(g\tau\sqrt{[n'+2]}_f\right)$$
$$\times\rho_F^{(K-1)}(n,n'+2)$$
$$+i|\rho_{ab}|\exp(-i\varphi)\cos\left(g\tau\sqrt{[n]_f}\right)\sin\left(g\tau\sqrt{[n']_f}\right)$$
$$\times\rho_F^{(K-1)}(n,n'-2)$$
$$-i|\rho_{ab}|\exp(-i\varphi)\sin\left(g\tau\sqrt{[n+2]_f}\right)\cos\left(g\tau\sqrt{[n'+2]_f}\right)$$
$$\times\rho_F^{(K-1)}(n+2,n')$$
$$-i|\rho_{ab}|\exp(i\varphi)\sin\left(g\tau\sqrt{[n]_f}\right)\cos\left(g\tau\sqrt{[n']_f}\right)$$
$$\times\rho_F^{(K-1)}(n-2,n'). \quad (35)$$

To solve the above equation we employ the same method as in previous section. A moment's inspection of the equation (35) shows that it is convenient to propose the phase-independent matrix elements $\tilde{\rho}^{(K)}(n,n')$ through the definitions

$$\rho_F^{(K)}(n,n') = (i\exp(-i\varphi))^{(n'-n)/2}\Bigg(\prod_{\ell=1}^{n/2}\sin\left(g\tau\sqrt{[2\ell]_f}\right)$$
$$\times\prod_{\ell'=1}^{n'/2}\sin\left(g\tau\sqrt{[2\ell']_f}\right)\Bigg)\rho_{aa}^{(n+n')/4}\tilde{\rho}^{(K)}(n,n') \quad (36a)$$

for even n,n',

$$\rho_F^{(K)}(n,n') = (i\exp(-i\varphi))^{(n'-n)/2}$$
$$\times\Bigg(\prod_{\ell=1}^{(n-1)/2}\sin\left(g\tau\sqrt{[2\ell+1]_f}\right)$$
$$\times\prod_{\ell'=1}^{(n'-1)/2}\sin\left(g\tau\sqrt{[2\ell'+1]_f}\right)\Bigg)$$
$$\times\rho_{aa}^{(n+n')/4}\tilde{\rho}^{(K)}(n,n') \quad (36b)$$

for odd n,n',

$$\rho_F^{(K)}(n,n') = (i\exp(-i\varphi))^{(n'-n)/2}\Bigg(\prod_{\ell=1}^{n/2}\sin\left(g\tau\sqrt{[2\ell]_f}\right)$$
$$\times\prod_{\ell'=1}^{(n'-1)/2}\sin\left(g\tau\sqrt{[2\ell'+1]_f}\right)\Bigg)\rho_{aa}^{(n+n')/4}\tilde{\rho}^{(K)}(n,n'), \quad (36c)$$

for even n and odd n' and

$$\rho_F^{(K)}(n,n') = (i\exp(-i\varphi))^{(n'-n)/2}$$
$$\times\Bigg(\prod_{\ell=1}^{(n-1)/2}\sin\left(g\tau\sqrt{[2\ell+1]_f}\right)\prod_{\ell'=1}^{n'/2}\sin\left(g\tau\sqrt{[2\ell']_f}\right)\Bigg)$$
$$\times\rho_{aa}^{(n+n')/4}\tilde{\rho}^{(K)}(n,n'), \quad (36d)$$

for odd n and even n'. By substituting the expressions (36) in (35) we find the following recursion relation for the matrix elements $\tilde{\rho}^{(K)}(n,n')$ which is valid for all values of n and n'

$$\tilde{\rho}^{(K)}(n,n') = \Big(\rho_{aa}\cos\left(g\tau\sqrt{[n+2]_f}\right)\cos\left(g\tau\sqrt{[n'+2]_f}\right)$$
$$+\rho_{bb}\cos\left(g\tau\sqrt{[n]_f}\right)\cos\left(g\tau\sqrt{[n']_f}\right)\Big)\tilde{\rho}^{(K-1)}(n,n')$$
$$+\rho_{aa}\rho_{bb}\sin^2\left(g\tau\sqrt{[n+2]_f}\right)\sin^2\left(g\tau\sqrt{[n'+2]_f}\right)$$
$$\times\tilde{\rho}^{(K-1)}(n+2,n'+2)+\tilde{\rho}^{(K-1)}(n-2,n'-2)$$
$$-\rho_{aa}\sqrt{\rho_{bb}}\cos\left(g\tau\sqrt{[n+2]_f}\right)\sin^2\left(g\tau\sqrt{[n'+2]}_f\right)$$
$$\times\tilde{\rho}^{(K-1)}(n,n'+2)$$
$$-\rho_{aa}\sqrt{\rho_{bb}}\sin^2\left(g\tau\sqrt{[n+2]_f}\right)\cos\left(g\tau\sqrt{[n'+2]}_f\right)$$
$$\times\tilde{\rho}^{(K-1)}(n+2,n')$$
$$+\sqrt{\rho_{bb}}\cos\left(g\tau\sqrt{[n]_f}\right)\tilde{\rho}^{(K-1)}(n,n'-2)$$
$$+\sqrt{\rho_{bb}}\cos\left(g\tau\sqrt{[n']_f}\right)\tilde{\rho}^{(K-1)}(n-2,n'), \quad (37)$$

with the initial conditions

$$\tilde{\rho}^{(0)}(n,n') = (i\exp(-i\varphi))^{(n-n')/2}\Bigg(\prod_{\ell=1}^{n/2}\sin\left(g\tau\sqrt{[2\ell]_f}\right)$$
$$\times\prod_{\ell'=1}^{n'/2}\sin\left(g\tau\sqrt{[2\ell']_f}\right)\Bigg)^{-1}\rho_{aa}^{-(n+n')/4}\rho_F^{(0)}(n,n') \quad (38a)$$

for even n, n',

$$\tilde{\rho}^{(0)}(n,n') = (i\exp(-i\varphi))^{(n-n')/2} \times \left(\prod_{\ell=1}^{(n-1)/2} \sin\left(g\tau\sqrt{[2\ell+1]_f}\right) \times \prod_{\ell'=1}^{(n'-1)/2} \sin\left(g\tau\sqrt{[2\ell'+1]_f}\right)\right)^{-1} \times \rho_{aa}^{-(n+n')/4}\rho_F^{(0)}(n,n') \quad (38b)$$

for odd n, n',

$$\tilde{\rho}^{(0)}(n,n') = (i\exp(-i\varphi))^{(n-n')/2} \left(\prod_{\ell=1}^{n/2} \sin\left(g\tau\sqrt{[2\ell]_f}\right) \times \prod_{\ell'=1}^{(n'-1)/2} \sin\left(g\tau\sqrt{[2\ell'+1]_f}\right)\right)^{-1} \rho_{aa}^{-(n+n')/4}\rho_F^{(0)}(n,n') \quad (38c)$$

for even n and odd n' and

$$\tilde{\rho}^{(0)}(n,n') = (i\exp(-i\varphi))^{(n-n')/2} \left(\prod_{\ell=1}^{(n-1)/2} \sin\left(g\tau\sqrt{[2\ell+1]_f}\right) \times \prod_{\ell'=1}^{n'/2} \sin(g\tau\sqrt{[2\ell']_f})\right)^{-1} \rho_{aa}^{-(n+n')/4}\rho_F^{(0)}(n,n'). \quad (38d)$$

for odd n and even n'.

Now, as before, we consider the case of weak atom-field interaction, that is,

$$g\tau \ll 1, \quad g\tau\sqrt{[\langle n\rangle]_f} \equiv g\tau\sqrt{\langle n\rangle(\langle n\rangle - 1)f^2(\langle n\rangle)} \ll 1. \quad (39)$$

In the first-order approximation the recursion relation (37) becomes

$$\tilde{\rho}^{(K)}(n,n') \approx \tilde{\rho}^{(K-1)}(n,n') + \tilde{\rho}^{(K-1)}(n-2,n'-2) + \sqrt{\rho_{bb}}\left(\tilde{\rho}^{(K-1)}(n,n'-2) + \tilde{\rho}^{(K-1)}(n-2,n')\right) \quad (40)$$

whose solution reads as

$$\tilde{\rho}^{(K)}(n,n') \approx \sum_{k,k'=0}^{K} \sum_{p=0}^{\min(k,k')} \frac{K!\rho_{bb}^{(k+k'-2p)/2}}{p!(k-p)!(k'-p)!(K-k-k'+p)!} \times \tilde{\rho}^{(0)}(n-2k, n'-2k'). \quad (41)$$

Following the same lines as those of the previous section we find that the micromaser field, initially prepared in a pure state $|\psi_F^{(0)}\rangle$, evolves into the pure state $|\psi_F^{(K)}\rangle$ as follows

$$\left\langle n \,\middle|\, \psi_F^{(K)}\right\rangle = \sum_{k=0}^{n} \frac{z^k}{k!\sqrt{[n-2k]_f!!}}\sqrt{[n]_f!!}\left\langle n-2k \,\middle|\, \psi_F^{(0)}\right\rangle, \quad (42)$$

where $z = -i\exp(-i\varphi)Kg\tau\sqrt{\rho_{aa}\rho_{bb}}$ and by definition

$$[n]_f!! = [2]_f[4]_f[6]_f...[n-2]_f[n]_f, \quad \text{for } n = 2k\ (k = 0,1,2,...),$$
$$[n]_f!! = [3]_f[5]_f[7]_f...[n-2]_f[n]_f, \quad \text{for } n = 2k+1\ (k = 0,1,2,...), \quad (43)$$

and $[0]_f!! = [1]_f!! = 1$.

Now let us assume that the cavity-field is initially in the vacuum state, i.e., $|\psi_F^{(0)}\rangle = |0\rangle$. Thus from (42) we find the following approximate expression for the normalized cavity-field state after passing K atoms ($K \gg 1$)

$$\left|\psi_F^{(K)}\right\rangle \equiv |z,0\rangle_f = C_0\sum_{n=0}^{\infty}\frac{z^n}{n!}\sqrt{(2n)!}f(2n)!!\,|2n\rangle, \quad (44a)$$

where by definition $f(2n)!! = f(2)f(4)\dots f(2n-2)f(2n)$ and C_0 is the normalization constant given by

$$C_0 = \left(\sum_{n=0}^{\infty}|z|^{2n}(2n)!(f(2n)!!)^2/(n!)^2\right)^{-1/2}. \quad (44b)$$

While if the cavity-field starts from the first excited state, $|\psi_F^{(0)}\rangle = |1\rangle$, then it evolves to the state

$$\left|\psi_F^{(K)}\right\rangle \equiv |z,1\rangle_f = C_1\sum_{n=0}^{\infty}\frac{z^n}{n!}\sqrt{(2n+1)!}f(2n+1)!!\,|2n+1\rangle, \quad (45a)$$

where by definition $f(2n+1)!! = f(3)f(5)...f(2n-1)f(2n+1)$ and C_1 is the normalization constant given by

$$C_1 = \left(\sum_{n=0}^{\infty}|z|^{2n}(2n+1)!(f(2n+1)!!)^2/(n!)^2\right)^{-1/2}. \quad (45b)$$

In order to get more clear insight to the nature of the states (44a) and (45a), we present the following argument. As stated before, there are two vacua for the operator $\hat{C}$, namely $|0\rangle$ and $|1\rangle$. Thus there are two infinite dimensional sectors, e.g. S_0 and S_1, corresponding to the states that are annihilated by $\hat{C}$. By applying the method of Shantha et al. [37] we can construct the operators $\hat{B}_0^+$ and $\hat{B}_1^+$ such that the commutators $[\hat{C},\hat{B}_0^+] = 1$ and $[\hat{C},\hat{B}_1^+] = 1$ hold in the sectors S_0 and S_1, respectively. For the sector S_0, constructed by repeatedly applying $\hat{C}^+$ on the ground state $|0\rangle$, we obtain

$$\hat{B}_0^+ = \frac{1}{2}(\hat{a}^+)^2\frac{1}{\hat{N}+1}\frac{1}{f(\hat{N}+2)}; \quad \left([\hat{C},\hat{B}_0^+] = 1, [\hat{B}_0,\hat{C}^+] = 1\right) \quad (46)$$

and for the sector S_1, constructed by repeatedly applying $\hat{C}^+$ on the first excited state $|1\rangle$, we obtain

$$\hat{B}_1^+ = \frac{1}{2}(\hat{a}^+)^2 \frac{1}{\hat{N}+2}\frac{1}{f(\hat{N}+2)};\quad \left([\hat{C},\hat{B}_1^+]=1,\quad [\hat{B}_1,\hat{C}^+]=1\right). \tag{47}$$

We now construct the two following displacement operators corresponding to the algebras $[\hat{B}_0,\hat{C}^+]=1$ and $[\hat{B}_1,\hat{C}^+]=1$, respectively

$$\hat{D}_f^{(0)}(z) = \exp(z\hat{C}^+ - z^*\hat{B}_0) = \exp(-|z|^2/2)\exp(z\hat{C}^+)\exp(-z^*\hat{B}_0), \tag{48a}$$

$$\hat{D}_f^{(1)}(z) = \exp(z\hat{C}^+ - z^*\hat{B}_1) = \exp(-|z|^2/2)\exp(z\hat{C}^+)\exp(-z^*\hat{B}_1). \tag{48b}$$

Applying $\hat{D}_f^{(0)}(z)$on $|0\rangle$ we obtain

$$\exp(z\hat{C}^+ - z^*\hat{B}_0)|0\rangle = e^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n}{n!}(f(\hat{N})(\hat{a}^+)^2)^n|0\rangle = e^{-|z|^2/2}\sum_{m=0}^{\infty}\frac{z^n}{n!}\sqrt{(2n)!}f(2n)!!\,|2n\rangle \tag{49}$$

which is the same as the state $|z,0\rangle_f$ up to a normalization constant. On the other hand applying the operator $\hat{D}_f^{(1)}(z)$ on the state $|1\rangle$ yields

$$\exp(z\hat{C}^+ - z^*\hat{B}_1)|1\rangle = e^{-|z|^{2/2}}\sum_{n=0}^{\infty}\frac{z^n}{n!}(f(\hat{N})(\hat{a}^+)^2)^n|1\rangle = e^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n}{n!}\sqrt{(2n+1)!}f(2n+1)!!\,|2n+1\rangle \tag{50}$$

which is the same as the state $|z,\,1\rangle_f$up to a normalization constant. Additionally, it is easy to verify that

$$\hat{B}_0|z,0\rangle_f = z|z,0\rangle_f,\quad \hat{B}_1|z,1\rangle_f = z|z,1\rangle_f. \tag{51}$$

So each of the two states $|z,0\rangle_f$ and $|z,1\rangle_f$ not only can be obtained by the application of a displacement type operator but also as a nonlinear (or f-deformed) annihilation operator eigenstate. In this manner each of these states can be interpreted as a type of nonlinear coherent states. Besides, in the limit $f(n)=1$ the structures of the states $|z,0\rangle_f$ and $|z,1\rangle_f$ are remindful of the usual squeezed vacuum and squeezed first excited states [38], respectively. Accordingly, it is reasonable to consider the states $|z,0\rangle_f$ and $|z,1\rangle_f$, respectively, as the nonlinear (f-deformed) squeezed vacuum and nonlinear (f-deformed) squeezed first excited states [39]. These two states correspond to the algebras $[\hat{B}_0,\hat{C}^+]=1$ and $[\hat{B}_1,\hat{C}^+]=1$, respectively.

Therefore, if the atom-field interaction is described by the two-photon intensity-dependent Jaynes-Cummings Hamiltonian (32), then under the conditions of no losses and weak atom-field interaction together with a large enough of polarized injected atoms two other families of NLCSs can be created.

Let us now consider the two cases in which the atom-field interaction is governed by the operators $(\hat{B}_0,\hat{B}_0^+)$ and $(\hat{B}_1,\hat{B}_1^+)$, respectively.

In the first case the corresponding interaction Hamiltonian is

$$\hat{H}_{ID}^{(m=2)} = g\left(\hat{B}_0|a\rangle\langle b| + |b\rangle\langle a|\hat{B}_0^+\right). \tag{52}$$

The operators $\hat{B}_0,\hat{B}_0^+$ and $\hat{N}$ satisfy the following closed algebraic relations

$$[\hat{B}_0,\hat{B}_0^+] = [[\hat{N}+2]]_f^{(0)} - [[\hat{N}]]_f^{(0)},\quad [\hat{N},\hat{B}_0] = -2\hat{B}_0,\quad [\hat{N},\quad \hat{B}_0^+] = 2\hat{B}_0^+, \tag{53a}$$

together with

$$\hat{B}_0|n\rangle = \sqrt{[[n]]_f^{(0)}}\,|n-2\rangle,\quad \hat{B}_0^+|n\rangle = \sqrt{[[n+2]]_f^{(0)}}\,|n+2\rangle, \tag{53b}$$

where the symbol $[[X]]_f^{(0)}$ stands for $(1/4)(X/(X-1))(1/f^2(X))$. By applying exactly the same procedure as before we find that the micromaser field, initially in the vacuum state $|0\rangle$, after passing a large number of polarized injected atoms evolves to the state

$$\left|\psi_F^{(K)}\right\rangle \equiv |z,0\rangle_f^{(e)} = C_e\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{(2n)!}f(2n)!!}|2n\rangle \tag{54a}$$

with

$$C_e = \left(\sum_{n=0}^{\infty}|z|^{2n}/(2n)!(f(2n)!!)^2\right)^{-1/2} \tag{54b}$$

as the normalization constant. Furthermore, it is easy to show that the state $|z,0\rangle_f^{(e)}$ can be constructed either by the action of the displacement operator $\hat{D}_f^{(0)'}(z) = \exp(z\hat{B}_0^+ - z^*\hat{C})$ on the ground state $|0\rangle$ or as the eigenstate of the operator $\hat{C}$ with the eigenvalue z.

We now turn to the second case. The corresponding interaction Hamiltonian reads as

$$\hat{H}_{ID}^{(m=2)} = g\left(\hat{B}_1|a\rangle\langle b| + |b\rangle\langle a|\hat{B}_1^+\right). \tag{55}$$

The operators $\hat{B}_1,\hat{B}_1^+$ and $\hat{N}$ satisfy the following closed algebraic relations

$$[\hat{B}_1,\hat{B}_1^+] = [[\hat{N}+2]]_f^{(1)} - [[\hat{N}]]_f^{(1)},\quad [\hat{N},\hat{B}_1] = -2\hat{B}_1,\quad [\hat{N},\hat{B}_1^+] = 2\hat{B}_1^+, \tag{56a}$$

together with

$$\hat{B}_1 |n\rangle = \sqrt{[[n]]_f^{(1)}}\, |n-2\rangle\,,\ \hat{B}_1^+ |n\rangle = \sqrt{[[n+2]]_f^{(1)}}\, |n+2\rangle, \tag{56b}$$

where the symbol $[[X]]_f^{(1)}$ stands for $(1/4)((X-1)/X)(1/f^2(X))$. This time we find that the micromaser field, initially prepared in the first excited state $|1\rangle$, after passing a large number of injected atoms evolves to the state

$$\left|\psi_F^{(K)}\right\rangle \equiv |z,1\rangle_f^{(o)} = C_o \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{(2n+1)!}f(2n+1)!!}\, |2n+1\rangle \tag{57a}$$

with

$$C_o = \left(\sum_{n=0}^{\infty} |z|^{2n}/(2n+1)!(f(2n+1)!!)^2\right)^{-1/2} \tag{57b}$$

as the normalization constant. In addition, it is found that the state $|z,1\rangle_f^{(o)}$ can be constructed either by the action of the displacement operator $\hat{D}_f^{(1)'}(z) = \exp(z\hat{B}_1^+ - z^*\hat{C})$ on the state $|1\rangle$ or as the eigenstate of the operator $\hat{C}$ with the eigenvalue z.

In fact, the states $|z,0\rangle_f^{(e)}$ and $|z,1\rangle_f^{(o)}$ are respectively the even and odd NLCSs which are defined [39,40] as the extensions of the notions of usual even and odd coherent states. Therefore we have shown the possibility of the generation of even and odd NLCSs of the radiation field in a coherently pumped micromaser where the atom-field interaction is governed by (52) and (55), respectively. Algebraically, the two states correspond to the algebras $[\hat{C}, \hat{B}_0^+] = 1$ and $[\hat{C}, \hat{B}_1^+] = 1$, respectively.

The normalizability condition leads to the following restrictions for the values of $|z|^2 = (Kg\tau)^2\rho_{aa}\rho_{bb}$ for the states $|z,0\rangle_f$, $|z,1\rangle_f$, $|z,0\rangle_f^{(e)}$ and $|z,1\rangle_f^{(o)}$,

$$|z,0\rangle_f: \quad |z|^2 < \lim_{m\to\infty} (m+1)/(4m+2)f^2(2m+2), \tag{58a}$$

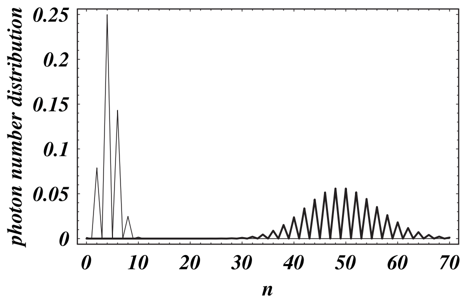
$$|z,1\rangle_f: \quad |z|^2 < \lim_{m\to\infty} (m+1)/(4m+6)f^2(2m+3), \tag{58b}$$

$$|z,0\rangle_f^{(e)}: \quad |z|^2 < \lim_{m\to\infty} (2m+2)(2m+1)f^2(2m+2), \tag{58c}$$

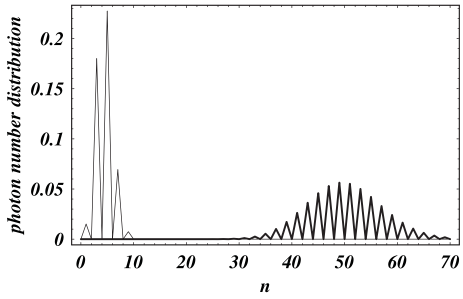
$$|z,1\rangle_f^{(o)}: \quad |z|^2 < \lim_{m\to\infty} (2m+3)(2m+1)f^2(2m+3). \tag{58d}$$

If $f(m)$ decreases (increases) faster than m^{-1} (m) for large m, then in the cases of $|z,0\rangle_f$ and $|z,1\rangle_f$ ($|z,0\rangle_f^{(e)}$ and $|z,1\rangle_f^{(o)}$) the range of $|z|^2$ is unrestricted.

In Figures 2a and 2b we plot the photon-number distribution for the states $|z,0\rangle_f$ and $|z,1\rangle_f$ respectively, with $g\tau = 10^{-3}$, $K = 10^4$, $\rho_{aa} = \rho_{bb} = 0.5$ and for two different cases $f(n) = 1/\sqrt{n+1}$ and $f(n) = 1/(n+1)$. (Note that for the above values of the parameters the states $|z,0\rangle_{f=1}$ and $|z,1\rangle_{f=1}$ are not normalizable.) As it is seen, both the nonlinear squeezed vacuum and nonlinear squeezed first excited states have oscillatory occupation number distribution. We also find



(a)



(b)

Fig. 2. Photon number distribution for the states $|z,0\rangle_f$ (a) and $|z,1\rangle_f$ (b) with $g\tau = 10^{-3}$, $K = 10^4$, $\rho_{aa} = \rho_{bb} = 0.5$ and for two different cases $f(n) = 1/\sqrt{n+1}$ (thick line) and $f(n) = 1/(n+1)$ (thin line).

$$\langle n\rangle_{|z,0\rangle_{f=1/\sqrt{n+1}}} = 48.484; \quad Q_{|z,0\rangle_{f=1/\sqrt{n+1}}} = 0.0316$$
(super-Poissonian statistics),

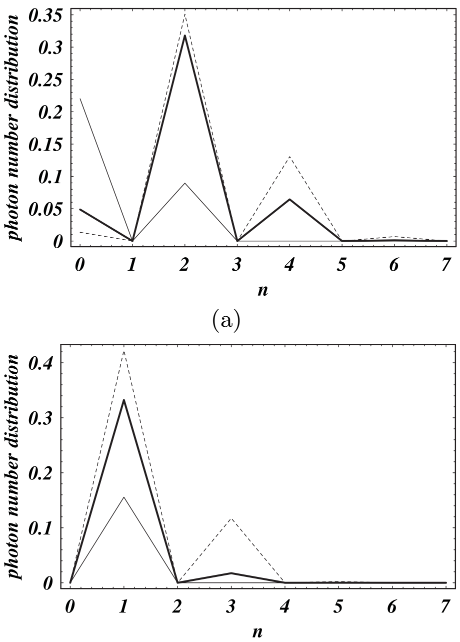
$$\langle n\rangle_{|z,0\rangle_{f=1/n+1}} = 3.352; \quad Q_{|z,0\rangle_{f=1/n+1}} = -0.219$$
(sub-Poissonian statistics),

$$\langle n\rangle_{|z,1\rangle_{f=1/\sqrt{n+1}}} = 50.980; \quad Q_{|z,1\rangle_{f=1/\sqrt{n+1}}} = -0.018$$
(sub-Poissonian statistics),

$$\langle n\rangle_{|z,1\rangle_{f=1/n+1}} = 4.543; \quad Q_{|z,1\rangle_{f=1/n+1}} = -0.453$$
(sub-Poissonian statistics).

Figures 3a and 3b, respectively, display the photon-number distribution for the states $|z,0\rangle_f^{(e)}$ and $|z,1\rangle_f^{(o)}$, with the same values of parameters as in Figures 1 and 2, and for $f(n) = 1$, $f(n) = \sqrt{n+1}$, $f(n) = n+1$. Furthermore, we obtain

$$\langle n\rangle_{|z,0\rangle_{f=1}^{(e)}} = 2.499; \quad Q_{|z,0\rangle_{f=1}^{(e)}} = -0.5$$
(sub-Poissonian statistics),



(a)

(b)

Fig. 3. Photon number distribution for the states $|z,0\rangle_f^{(e)}$ (a) and $|z,1\rangle_f^{(o)}$ (b) with $g\tau = 10^{-3}$, $K = 10^4$, $\rho_{aa} = \rho_{bb} = 0.5$ and for three different cases $f(n) = 1$ (------) $f(n) = \sqrt{n+1}$ (▬▬) and $f(n) = n+1$ (——).

$$\langle n\rangle_{|z,0\rangle^{(e)}_{f=\sqrt{n+1}}} = 2.058; \quad Q_{|z,0\rangle^{(e)}_{f=\sqrt{n+1}}} = -0.480$$

(sub-Poissonian statistics),

$$\langle n\rangle_{|z,0\rangle^{(e)}_{f=n+1}} = 0.478; \quad Q_{|z,0\rangle^{(e)}_{f=n+1}} = -0.370$$

(sub-Poissonian statistics),

$$\langle n\rangle_{|z,1\rangle^{(o)}_{f=1}} = 2; \quad Q_{|z,1\rangle^{(o)}_{f=1}} = -0.375$$

(sub-Poissonian statistics),

$$\langle n\rangle_{|z,1\rangle^{(o)}_{f=\sqrt{n+1}}} = 1.570; \quad Q_{|z,1\rangle^{(o)}_{f=\sqrt{n+1}}} = -0.340$$

(sub-Poissonian statistics),

$$\langle n\rangle_{|z,1\rangle^{(o)}_{f=n+1}} = 0.150; \quad Q_{|z,1\rangle^{(o)}_{f=n+1}} = -0.100$$

(sub-Poissonian statistics).

In the end, it is necessary to remark two important points. First, we note that the density matrix elements for the micromaser field states under consideration depend on the number of injected atoms, i.e., $\rho_F^{(K)}(n,n') = \langle n||\Psi\rangle_F^{(K)}{}_F^{(K)}\langle\Psi||n'\rangle \propto K^{n+n'}$. In particular, for $n = n'$ we have $\rho_F^{(K)}(n,n) \propto K^{2n}$ which is the hallmark of a cooperative process and is similar to the superradiance of a collective Dicke system [41]. Such dependence is due to the initial atomic coherence and indicates that emission of separately injected single atoms initially prepared in the same coherent superposition state is a cooperative process. As the second point, it should be noted that the micromaser field is unfortunately inaccessible for a direct measurement by a homodyning technique developed for optical fields. Therefore a usual way of its detection and measurement is to read a level statistics of a probe atom. By using this idea it has been recently presented [42] a new method for the determination of the intra-cavity field state, based on an operational definition of the Wigner function. We believe that this method can be employed, in principle, as an efficient technique to measure the properties of the NLCSs of the micromaser field generated following the procedure discussed in the present paper.

4 Summary

In this paper we have made an effort to solve a complicated dynamical problem characterized by a quite general nonlinear atom-field interaction Hamiltonian which naturally lends itself to the use of NLCSs. Our main work is investigating how to produce generic NLCSs (f-deformed CSs) in a lossless micromaser cavity which is pumped by a stream of velocity-selected two-level atoms prepared in a coherent superposition of the upper and lower states. Considering the intensity-dependent interaction, we have studied the quantum evolution of the cavity-field coupled to the polarized injected atoms through one as well as two-photon transitions. In the case of one-photon transitions, two different families of NLCSs can be generated if the cavity-field starts from the vacuum state. These families are associated with two dual deformed boson oscillator algebras. On the other hand, we have found that in the case of two-photon transitions four different families of NLCSs including nonlinear squeezed vacuum state, nonlinear squeezed first excited state, nonlinear even coherent state and nonlinear odd coherent state, can be generated. We have shown that the nonlinear squeezed vacuum state and nonlinear squeezed first excited state can be algebraically interpreted as the dual families of the even nonlinear coherent state and odd nonlinear coherent state, respectively.

The quantum statistical properties of the NLCSs we have obtained in our treatment depend on the form of nonlinearity function $f(n)$ as well as some physical parameters, i.e., g, τ and K. For example, it is found that they may demonstrate sub-Poissonian statistics so they can be regarded as non-classical states. Moreover, the density matrix elements of the states depend on the number of injected atoms. This dependence displays a feature of a cooperative process that is similar to superradiance.

Finally, we again stress that the presence of initial atomic coherence, weakness of atom-field interaction and largeness of passed atoms through the lossless cavity are some of the essential assumptions in our treatment.

References

1. E. Schrödinger, Die Naturwissenschaften **26**, 664 (1926)
2. R.J. Glauber, Phys. Rev. Lett. **70**, 84 (1963); R.J. Glauber, Phys. Rev. **130**, 2529 (1963); R.J. Glauber, Phys. Rev. **131**, 2766 (1963)
3. A. Perelomov, *Generalized Coherent States and Their Applications* (Springer, Berlin, 1986)
4. J.R. Klauder, J. Math. Phys. **4**, 1055 (1963)
5. M.H. Nieto, L.M. Simmons Jr., Phys. Rev. Lett. **41**, 207 (1978)
6. L.C. Biedenharn, J. Phys. A: Math. Gen. **22**, L873 (1989); A.J. Macfarlane, J. Phys. A: Math. Gen. **22**, 4581 (1989)
7. R.L. de Matos Filho, W. Vogel, Phys. Rev. A **54**, 4560 (1996)
8. V.I. Manko, G. Marmo, E.C.G. Sudarshan, F. Zaccaria, Phys. Scripta **55**, 528 (1997)
9. E.C. Lerner, H.W. Huang, G.E. Walters, J. Math. Phys. **11**, 1679 (1970)
10. Y. Aharonov, H.W. Huang, J.M. Knight, E.C. Lerner, Lett. Nuovo Cim. **2**, 1317 (1971)
11. G.S. Agarwal, K. Tara, Phys. Rev. A **43**, 492 (1991)
12. B. Roy, Phys. Lett. A **249**, 25 (1998); X.G. Wang, H.C. Fu, Mod. Phys. Lett. B **13**, 617 (1999); X.G. Wang, Opt. Commun. **178**, 365 (2000)
13. D. Meschede, H. Walther, G. Muller, Phys. Rev. Lett. **54**, 551 (1985); P. Filipowicz, J. Javanainen, P. Meystre, Phys. Rev. A **34**, 3077 (1986); F. Diedrich, J. Krause, G. Rempe, M.O. Scully, H. Walther, IEEE J. Quant. Electron. **24**, 1314 (1988); S. Haroche, J.M. Raimond, *Advances in Atomic and Molecular Physics*, edited by D.R. Bates, B. Bederson (Academic Press, New York, 1985), Vol. 20, p. 350; G. Rempe, F. Schmidt-Kaler, H. Walther, Phys. Rev. Lett. **64**, 2783 (1990)
14. J. Krause, M.O. Scully, H. Walther, Phys. Rev. A **36**, 4547 (1987)
15. L.A. Lugiato, M.O. Scully, H. Walther, Phys. Rev. A **36**, 740 (1987)
16. S. Qamar, K. Zaheer, M.S. Zubairy, Opt. Commun. **78**, 341 (1990); S. Qamar, M.S. Zubairy, Phys. Rev. A **48**, 1559 (1993); Fam le Kein, G.M. Meyer, M.O. Scully, H. Walther, Shi-Yao Zhu, Phys. Rev. A **49**, 1367 (1994)
17. P. Meystre, G. Rempe, H. Walther, Opt. Lett. **13**, 1078 (1988)
18. J.J. Slosser, P. Meystre, Phys. Rev. A **41**, 3867 (1990)
19. E.T. Jaynes, F.W. Cummings, Proc. IEEE **51**, 89 (1963)
20. B. Buck, C.V. Sukumar, Phys. Lett. A **81**, 132 (1981)
21. V. Buzek, Phys. Rev. A **39**, 3196 (1989)
22. M. Chaichian, D. Ellinas, P. Kulish, Phys. Rev. Lett. **65**, 980 (1990)
23. T. Kato, *Perturbation Theory for Linear Operators* (Springer, Berlin, 1966)
24. A. Rosenhouse, J. Mod. Opt. **38**, 269 (1991)
25. D. Bonatsos, C. Daskaloyannis, G. Lalazissis, Phys. Rev. A **47**, 3448 (1993)
26. B.W. Shore, P. Knight, J. Mod. Opt. **40**, 1195 (1993)
27. Zh. Tang, Phys. Rev. A **52**, 3448 (1995)
28. A. Messina, S. Maniscalco, A. Napoli, J. Mod. Opt. **50**, 1 (2003)
29. M. Abdel-Aty, Physica A **313**, 471 (2002); M. Abdel-Aty, J. Phys. A: Math. Gen. **37**, 1759 (2004)
30. M.H. Naderi, M. Soltanolkotabi, R. Roknizadeh, J. Phys. Soc. Jap. **73**, 2413 (2004)
31. C.J. Hood, S. Chapman, T.W. Lynn, H.J. Kimble, Phys. Rev. Lett. **80**, 4157 (1998); P. Munstermann, T. Fischer, P. Maunz, P.W.H. Pinske, G. Rempe, Opt. Commun. **159**, 63 (1999); P. Munstermann, T. Fischer, P. Maunz, P.W.H. Pinske, G. Rempe, Phys. Rev. Lett. **82**, 3791 (1999)
32. M. Soltanolkotabi, M.H. Naderi, Fizika A **10**, 1 (2001)
33. F.L. Kien, M.O. Scully, H. Walther, Found. Phys. **23**, 177 (1993)
34. W.H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973)
35. L. Davidovich, J.M. Raimond, M. Brune, S. Haroche, Phys. Rev. A **36**, 3771 (1987)
36. M. Brune, J.M. Raimond, P. Goy, L. Davidovich, S. Haroche, Phys. Rev. Lett. **59**, 1899 (1987)
37. P. Shanta, S. Chaturvedi, V. Srinivasan, G.S. Agarwal, C.L. Mehta, Phys. Rev. Lett. **72**, 1447 (1994)
38. D. Stoler, Phys. Rev. D **4**, 2309 (1971); H.P. Yuen, Phys. Rev. A **13**, 2226 (1976)
39. S. Sivakumar, J. Opt. B: Quant. Semiclass. Opt. **2**, R61 (2000)
40. S. Mancini, Phys. Lett. A **233**, 291 (1997); S. Sivakumar, Phys. Lett. A **252**, 257 (1998); S. Sivakumar, J. Phys. A: Math. Gen. **33**, 2289 (2000)
41. M. Gross, S. Haroche, Phys. Rep. **93**, 301 (1982)
42. P. Lougovski, E. Solano, Z.M. Zhang, H. Walther, H. Mack, W.P. Schleich, Phys. Rev. Lett. **91**, 010401 (2003)